MATH2050C Selected Solution to Assignment 8

Section 4.2

(11c). Let $x_n = 1/(2n\pi) \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \operatorname{sgn} \sin 1/x_n = \lim_{n\to\infty} \operatorname{sgn} 0 = 0 \ .$$

On the other hand, let $y_n = 1/(2n\pi + \pi/2) \to 0$ as $n \to \infty$. We have

$$\lim_{n\to\infty} \operatorname{sgn} \sin 1/y_n = \lim_{n\to\infty} \operatorname{sgn}\ 1 = 1.$$

We conclude from Sequential Criterion that the limit does not exist.

(11d). Using the inequality

$$|\sqrt{x} \sin 1/x^2| \le \sqrt{x} \ ,$$

and the fact that $\lim_{x\to 0^+} \sqrt{x} = 0$, we conclude from Squeeze Theorem that

$$\lim_{x\to 0^+} \sqrt{x} \sin \frac{1}{x^2} = 0 \ .$$

Section 4.3

(3) Let $M > 0$ be given. We take $\delta = 1/M^2$. Then for $x, 0 < |x| < \delta$,

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{\delta}} = M \ ,$$

so $\lim_{x\to 0^+} 1/\sqrt{|x|} = \infty$.

(5a) The right hand limit is ∞. (More precisely, the right hand limit does not exist; it diverges to ∞.) Let $M > 0$. Choose $\delta = 1/M$. Then for $x, 1 < x < 1 + 1/M$,

$$\frac{x}{x-1} \ge \frac{1}{x-1} > M \ .$$

(5b) The limit does not exist. The right limit diverges to ∞ and the left limit diverges to $-\infty$. For $M > 0$. Choose $\delta = \min\{1/2, 1/(2M)\}$. Then for $x, 0 < x - 1 < \delta$, we have $x > 1 - \delta \ge 1 - 1/2 = 1/2$. Thus,

$$\frac{x}{x-1} \ge \frac{1/2}{x-1} > \frac{1}{2\delta} \ge \frac{1}{2 \times 1/(2M)} = M \ ,$$

which shows that $\lim_{x\to 1^+} x/(x-1) = \infty$. Similarly, one can show that $\lim_{x\to 1^-} x/(x-1) = -\infty$.

(5h) The limit exists and is equal to -1. For $x > 0$,

$$0 \le \left| \frac{\sqrt{x} - x}{\sqrt{x} + x} - (-1) \right| = \left| \frac{2\sqrt{x}}{\sqrt{x} + x} \right| \le \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \ .$$

By Squeeze Theorem,

$$0 \le \lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} \le \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \ .$$

Supplementary Problems

Justify your answers in the following problems.

1. Find the limits of x^3e^{-x} where $c = -\infty, 0$ and ∞ as $x \to c$.

   Solution. Observing that $\lim_{x\to-\infty} x^3e^{-x} = \lim_{x\to\infty} -x^3e^x$ and $e^x > 1$ for $x > 0$, we have $-x^3e^x < -x^3$, so $\lim_{x\to-\infty} x^3e^{-x} = -\infty$. Next, as $x^3 \to 0$ and $e^x \to e^0 = 1$ as $x \to 0$, by the Limit Theorem $\lim_{x\to 0} x^3e^x = \lim_{x\to 0} x^3 \lim_{x\to 0} e^x = 0 \times 1 = 0$. Finally, using $e^x \geq x^4/24$, $x^3e^{-x} \leq 24/x$ for $x > 0$. By Squeezing, $\lim_{x\to\infty} x^3e^{-x} = 0$.

3. Show that $x - \frac{x^3}{6} \leq \sin x \leq x$, for $x \in [0, 1]$ and deduce $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

   Solution. For $x \in [0, 1]$,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots \\
&= x - \left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) - \left(\frac{x^7}{7!} - \frac{x^9}{9!}\right) + \cdots \\
&\leq x\ ,
\end{aligned}$$

   since the terms in brackets are nonnegative. On the other hand,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots \\
&= x - \frac{x^3}{3!} + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \left(\frac{x^9}{9!} - \frac{x^11}{11!}\right) + \cdots \\
&\geq x - \frac{x^3}{6}\ ,
\end{aligned}$$

   since the terms in brackets are nonnegative. From $x - x^3/6 \leq \sin x \leq x, x \in [0, 1]$, we get $1 - x^2/6 \leq \sin x/x \leq 1$ for $x \in (0, 1]$. Since $\sin x/x$ is even, this inequality holds for $x \in [-1, 0)$ too. By Squeezing we conclude $\lim_{x\to 0} \sin x/x = 1$.

4. Find $\lim_{x\to 0} \sin 6x/ \sin 5x$.

   Solution. By the limit theorem and the previous problem

$$\lim_{x\to 0} \frac{\sin 6x}{\sin 5x} = \frac{6}{5} \lim_{x\to 0} \frac{\sin 6x}{6x} \lim_{x\to 0} \frac{5x}{\sin 5x} = \frac{6}{5}\ .$$

5. Find the limit of $\sqrt{(x + a)(x + b)} - x$ as $x \to \infty$. Here $a, b > 0$.

Solution. We have

$$\begin{aligned}
\sqrt{(x+a)(x+b)} - x &= \frac{(x+a)(x+b)-x^2}{\sqrt{(x+a)(x+b)}+x} \\
&= \frac{(a+b)x+ab}{\sqrt{(x+a)(x+b)}+x} \\
&= \frac{a+b+ab/x}{\sqrt{(1+a/x)(1+b/x)}+1} \\
&\to \frac{a+b}{2} \ ,
\end{aligned}$$

as $x \to \infty$.

6. Evaluate

$$\lim_{x\to -3} \frac{x^2-2x-15}{x+3}.$$

Solution. Use $x^2-2x-15 = (x+3)(x-5)$, we have $\frac{x^2-2x-15}{x+3} = \frac{(x+3)(x-5)}{x+3} = x-5$ as long as $x+3 \neq 0$. Therefore,

$$\lim_{x\to -3} \frac{x^2-2x-15}{x+3} = \lim_{x\to -3}(x-5) = -8.$$

7. Evaluate

$$\lim_{x\to\infty} \frac{\cos 1/x}{x}.$$

Solution. Observe that $|\cos 1/x| \le 1$ for all x. For $\varepsilon > 0$, take $K = 1/\varepsilon$. Then

$$\left|\frac{\cos 1/x}{x}\right| \le \frac{1}{x} < \varepsilon \ , \quad \forall x > K.$$

We conclude that the limit is equal to 0.

8. Find $\lim_{x\to c} \dfrac{5x-\sqrt{x}}{\sqrt{x}-x^3}$ for $c = 0^+$ and ∞.

Solution (a)

$$\lim_{x\to 0^+} \frac{5x-\sqrt{x}}{\sqrt{x}-x^3} = \lim_{x\to 0^+} \frac{5\sqrt{x}-1}{1-x^{5/2}} = \frac{\lim_{x\to 0^+}(5\sqrt{x}-1)}{\lim_{x\to 0^+}(1-x^{5/2})} = -1 \ .$$

(b) For $x \ge 1$,

$$\left|\frac{5x-\sqrt{x}}{\sqrt{x}-x^3}\right| = \left|\frac{5\sqrt{x}-1}{1-x^{5/2}}\right| \le \frac{5\sqrt{x}}{x^{5/2}-1} = \frac{5}{x^2}\frac{1}{1-x^{-5/2}} \ .$$

As

$$\lim_{x\to\infty} \frac{5}{x^2}\frac{1}{1-x^{-5/2}} = 0 \ ,$$

by Squeeze Theorem

$$\lim_{x\to\infty} \frac{5x-\sqrt{x}}{\sqrt{x}-x^3} = 0 \ .$$